Family of Gauss-Seidel Method for Solving 2D Reynolds Equation in Hydrodynamic Lubrication Problem

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Abstract

The purpose of this paper is to analyze the effectiveness of several Gauss-Seidel (GS) methods in solving 2D Reynolds equation in hydrodynamic lubrication problem. The governing partial differential equations were discretized using second order finite difference formulae. The resulting nonlinear systems were then solved iteratively using Full Sweep Gauss Seidel (FSGS), Half Sweep Gauss Seidel (HSGS) and Quarter Sweep Gauss Seidel (QSGS) methods. The source codes were written using Scilab 5.5 which is an open source software suitable for scientific computing. Finally a number of numerical experiments were conducted. Therefore we concluded that QSGS method is superior to the others.

Keywords. Finite difference methods; Iterative technique; Industrial computing; Reynolds equation; Hydrodynamic lubrication.

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1 Introduction

Bearings such as journal bearings and slider bearings have wide application in various industries such as machinery and equipment, automotive and aerospace. Analysis of these bearings is most importantly occurring in classical hydrodynamic lubrication theory and it is also most difficult and complex to solve due to integration of the Reynolds equation. Different numerical techniques for solving Reynolds equation have been used by many researchers. Sivak and Sivak [1] obtained a numerical solution of the Reynolds equation by modified Ritz method. Tayal et al. [2] used finite element methods (FEM) to investigate the effect of nonlinearity on the performance of journal bearings. Chandrawat and Sinhasan [3] presented a comparison between the Gauss-Seidel iterative method and the linear complementarity approach for determining the pressure field in the analysis of plain and two-axial groove journal bearings in laminar flow operation. Gero and Ettles [4] do a comparison of finite difference methods (FDM) and FEM in solving 1D and 2D Reynolds equation. Their results for two-dimensional bearings demonstrated that the relative errors of the FDM solutions were smaller than those associated with the FEM approach. Furthermore, it was shown that the FDM approach converged more rapidly than the FEM technique, with an average CPU time of 0.15 s as compared to 0.17 s for the FEM method. More recently, Gao et al. [5] used a CFD approach to analyze the performance of plain journal bearing under hydrodynamic lubrication by water. But computing programs by CFD are generally time consuming and not everyone has access to CFD software because of the cost involved. Obtaining the analytical solution for 2D Reynolds equation is difficult and almost impossible. But several works on approximate analytical solution was reported by Vignolo et al [6] using a regular perturbation method and Santos [7] et al. using a hybrid numerical-analytical solution known as Generalized Integral Transform Technique (GITT).

The concept of half-sweep iterative method was introduced by Abdullah [8] in solving two-dimensional Poisson equations. Applications of this concept in various partial differential equations were further investigated in [9-13]. Othman and Abdullah [14] also solved two-dimensional Poisson equations by introducing the concept of quarter-sweep iteration via the Modified Explicit Group (MEG) iterative methods. Again, applications of this concept have been intensively discussed in [15-18]. These two concepts (HSGS and QSGS) are essential to reduce the computational complexities during the iterative process, because the implementation of the half- and quarter-sweep iterations will only consider nearly half and quarter of all node points in a solution domain respectively. These have been shown by Ng and Hassan [19] when solving two-dimensional Poisson equation with Dirichlet boundary conditions. In this paper, we discretized the two dimensional Reynolds equation using finite difference methods and examined the applications of half-sweep Gauss Seidel (HSGS) and quarter sweep Gauss Seidel (QSGS) iterations method in solving the resultant nonlinear systems of equations. The standard GS iterative method also known as the Full-Sweep Gauss-Seidel iterative method is implemented as control method in order to examine the performance of HSGS and QSGS iterative methods.
2 Discretization technique

The problem considered here is based on the steady, incompressible lubrication problem [4, 20, 21]. The governing equation is Reynolds equation which the solution of this equation gives the pressure distribution inside the bearings; see Figure 1.

\[
\frac{\partial}{\partial x}(h^3 \frac{\partial p}{\partial x}) + \frac{\partial}{\partial y}(h^3 \frac{\partial p}{\partial y}) = 6 \nu \mu \frac{\partial h}{\partial x}
\]  

(1)

where \( p \) = pressure, \( h \) = film thickness, \( x, y \) = flow direction, \( V \) = flow direction, \( \nu \) = sliding velocity, \( \mu \) = oil viscosity.

Figure 1: Physical configuration of a finite journal bearing.

It is always convenient to use the dimensionless form of (1). For a journal bearing, the following parameters were used.

\[
\hat{X} = R \hat{\theta}, \hat{Y} = \frac{y}{b}, \hat{V} = R \omega, \hat{P} = \frac{p h^2}{6 \nu \nu}, \hat{H} = \frac{h}{c} = (1 + \epsilon \cos \hat{\theta}).
\]

(2)

Where \( \epsilon = R_1 - R_2 \) is the clearance between the radii of the bearing and the journal. The dimensionless two-dimensional Reynolds equation for journal bearing is then given by;

\[
\frac{\partial}{\partial \hat{\theta}}(H^3 \frac{\partial \hat{P}}{\partial \hat{\theta}}) + \alpha \frac{\partial}{\partial \hat{Y}}(H^3 \frac{\partial \hat{P}}{\partial \hat{Y}}) = -\epsilon \sin \hat{\theta} ; \alpha = \left( \frac{R}{h} \right)^2
\]

(3)

The boundary conditions are \( \hat{P}(\hat{X}, 0) = \hat{P}(\hat{X}, b) = \hat{P}(0, \hat{Y}) = \hat{P}(l, \hat{Y}) = 0 \) and \( \frac{\partial \hat{P}}{\partial \hat{X}} = 0 \) at the outlet.

We discretized the equations using 5 point stencil finite difference schemes as in Figure 2.
The following central finite difference equations for first and second derivatives were derived from Taylor’s series and have truncation error of $O(\Delta x^2)$.

\[
\frac{\partial \bar{P}}{\partial \bar{X}} = \frac{\bar{P}(i+1,j) - \bar{P}(i-1,j)}{2\Delta x} \tag{4}
\]

\[
\frac{\partial^2 \bar{P}}{\partial \bar{X}^2} = \frac{\bar{P}(i+1,j) - 2\bar{P}(i,j) + \bar{P}(i-1,j)}{(\Delta x)^2} \tag{5}
\]

Derivatives against $y$ will resemble (4) and (5). Substitute the finite difference form into (3):

\[
\frac{3}{\bar{H}} \frac{\partial}{\partial \bar{X}} \left( \frac{\bar{P}(i+1,j) - \bar{P}(i-1,j)}{2\Delta x} \right) + \alpha \left( \frac{3}{\bar{H}} \frac{\partial}{\partial \bar{Y}} \left( \frac{\bar{P}(i,j+1) - \bar{P}(i,j-1)}{2\Delta y} \right) + \frac{\bar{P}(i,j+1) - 2\bar{P}(i,j) + \bar{P}(i,j-1)}{(\Delta y)^2} \right) = \frac{1}{\bar{H^3}} \frac{\partial \bar{H}}{\partial \bar{X}} \tag{6}
\]

Simplifying (6), we get:

\[
\bar{H} = \frac{h}{c} = (1 + \varepsilon \cos \bar{\theta}) \Rightarrow \frac{\partial \bar{H}}{\partial \bar{X}} = -\varepsilon \sin \bar{\theta} \tag{7}
\]

According to [19], equation (8) can be generally written as:

\[
\bar{P}(i,j) = \frac{1}{A} \left[ \bar{P}(i-1,j)B + \bar{P}(i+1,j)C + \bar{P}(i,j-1)D + \bar{P}(i,j+1)E \right] + F \tag{8}
\]

Where

\[
G_1 = \frac{1}{(\Delta x)^2}; \quad G_2 = \frac{1}{(\Delta y)^2}; \quad A = 2(G_1 + G_2); \quad B = -\frac{3}{\bar{H}} \frac{\partial}{\partial \bar{X}} \frac{1}{2\Delta x} + G_1; \quad C = G_1 - B; \quad D = -\frac{3}{\bar{H}} \frac{\partial}{\partial \bar{Y}} \frac{1}{2\Delta y} + G_2; \quad E = G_2 - D; \quad F = \frac{1}{\bar{H^3}} \frac{\partial \bar{H}}{\partial \bar{X}} \tag{9}
\]
When $\phi = 1$ the formula gives us the FSGS method and QSGS when $\phi = 2$. Figure 3 shows the rotated 5-point FD scheme which is known as half-sweep Gauss-Seidel (HSGS) [19]. Applying the mesh to (10) we get:

$$
\bar{P}(i,j) = \frac{1}{A} \left[ \bar{P}(i - 1, j + 1)B + \bar{P}(i + 1, j + 1)C + \bar{P}(i + 1, j - 1)D + \bar{P}(i - 1, j - 1)E + F \right]
$$

Equation (11) defined the HSGS method with the distance between the points becomes $\beta = \sqrt{(\Delta x)^2 + (\Delta y)^2}$.

![Figure 3: Rotated 5-point finite difference scheme](image)

### 3 Results and Discussion

The discretized form of Reynolds equation in (11) was solved using FSGS, HSGS and QSGS iterative methods. The experiments were carried out on Intel Quadcore 2.7 GHz CPU with 2GB RAM using Scilab 5.5 programming software.

Convergence is achieved when the successive error is less than a specified tolerance which is $\text{epsilon} = 10^{-8}$. Effectiveness of the methods was accessed using the number of iterations and execution time. Table 1 clearly shows that QSGS iterative method is the fastest amongst the tested methods. Percentage of reduction in number of iteration and execution time is shown in Table 2.

### 4 Conclusion

Overall, the numerical experiments prove that QSGS is more effective compared to FSGS and HSGS in terms of number of iterations and execution time when solving the 2D Reynolds equation. In future our study will extend to successive over relaxation (SOR) and accelerated over relaxation (AOR) in solving more complex hydrodynamic lubrication problem.
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Table 1: Performance Comparison Between the Methods (L/D = 1, e = 0.7)

<table>
<thead>
<tr>
<th>Mesh Size MxN</th>
<th>Method</th>
<th>Number of Iteration</th>
<th>Execution Time (Seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>55</td>
<td>FSGS</td>
<td>30</td>
<td>0.041</td>
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<tr>
<td></td>
<td>HSGS</td>
<td>18</td>
<td>0.028</td>
</tr>
<tr>
<td></td>
<td>QSGS</td>
<td>14</td>
<td>0.015</td>
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<tr>
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<td>FSGS</td>
<td>118</td>
<td>0.779</td>
</tr>
<tr>
<td></td>
<td>HSGS</td>
<td>77</td>
<td>0.457</td>
</tr>
<tr>
<td></td>
<td>QSGS</td>
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</tr>
<tr>
<td>861</td>
<td>FSGS</td>
<td>437</td>
<td>11.283</td>
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<tr>
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<tr>
<td></td>
<td>QSGS</td>
<td>1201</td>
<td>293.35</td>
</tr>
</tbody>
</table>

Table 2: Reduction Percentages For The Number Of Iterations And Execution Time Of HSGS And QSGS Compared To FSGS

<table>
<thead>
<tr>
<th>Method</th>
<th>Number of Iterations (%)</th>
<th>Execution Time (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>HSGS</td>
<td>40 – 82.7</td>
<td>31.71 – 89.64</td>
</tr>
<tr>
<td>QSGS</td>
<td>53.33 – 89.92</td>
<td>63.41 – 92.26</td>
</tr>
</tbody>
</table>
Figure 4: Number of iterations for different mesh sizes

Figure 5: Execution time for different mesh sizes

References